

The Separation Theorem: Interest as a Removable Parameter in No-Arbitrage Pricing

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Abstract

The κ -rate programme rests on a single claim: that the interest rate is a *separable and removable* parameter of no-arbitrage pricing, leaving behind a convergence intensity κ that is a hazard, not a discount. This paper gives that claim its rigorous statement and proof, and develops the stochastic machinery it requires. We work in the *event parametrisation* of the perpetual price of Akerer et al. [2025]: an expectation of a payoff at a Cox (doubly-stochastic) stopping time θ with intensity κ , discounted at the interest parameter ι — the form that holds the real-event intensity fixed while the pure time-charge varies, and coincides with AHJ’s funding parametrisation exactly at the *riba-free* point $\iota = 0$ (away from it, AHJ’s ι enters both the discount and the effective stopping intensity; see §3). Our central result resolves the oldest objection to interest-free pricing—that setting the rate to zero makes prices diverge. We show this is false for the κ -priced operator and true only for the conventional one, and we locate the difference exactly: as a function of ι , the pricing operator is holomorphic on $\{\text{Re}(\iota) > -\kappa\}$ with its only singularity a simple pole at $\iota = -\kappa$; hence for $\kappa > 0$ the point $\iota = 0$ is interior and regular, and the *riba-free* price $f(0) = \mathbb{E}^\mathbb{Q}[X_\theta]$ is finite and well defined. The conventional perpetual operator $V(r) = c/r$ has its pole at $r = 0$, which is precisely why interest there is not removable. The convergence intensity $\kappa > 0$ moves the pole off the origin; that displacement *is* the Separation Theorem. We then prove exact multiplicative separation of ι from κ in the event parametrisation, the formal isomorphism with the Duffie–Singleton defaultable claim ($\kappa \leftrightarrow \lambda$, $\iota \leftrightarrow r$ here; equivalently $\kappa_{\text{AHJ}} - \iota \leftrightarrow \lambda$ in the funding parametrisation, the map of the credit-equivalence paper), specify κ as a CIR diffusion and carry it through a change of measure that preserves the Feller condition, derive the affine κ -term-structure, and obtain a corollary identifying the empirically measured risk-premium share with $|\lambda|/\alpha$. The κ -priced monetary base appears as a closing application.

Keywords: separation theorem, convergence intensity, Cox process, removable singularity, Cox–Ingersoll–Ross, change of measure, affine term structure, *riba*.

1 Setup

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ be a filtered probability space under the pricing (risk-neutral) measure $\mathbb{Q}$, satisfying the usual conditions, and let $X = (X_t)$ be the adapted spot price of an underlying asset.

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Assumption 1 (Cox convergence time). The convergence intensity $(\kappa_t)_{t \geq 0}$ is non-negative and adapted, and the convergence (or default) time θ is the first jump of a Cox process directed by κ : $\theta = \inf\{t : \int_0^t \kappa_s ds \geq E\}$ with $E \sim \text{Exp}(1)$ independent of $\mathcal{F}_\infty$, so that $\mathbb{Q}(\theta > t \mid \mathcal{F}_\infty^{X, \kappa}) = \exp(-\int_0^t \kappa_s ds)$. We assume $\mathbb{E}^\mathbb{Q}[X_\theta] < \infty$.

Definition 1 (The pricing operator). For an interest parameter $\iota \geq 0$ the no-arbitrage value of the perpetual claim paying X_θ at θ is

$$f(\iota) = \mathbb{E}^\mathbb{Q}[e^{-\iota\theta} X_\theta]. \quad (1)$$

A remark on parametrisation is owed here, because it is where a careless statement of separation would overreach. Equation (1) is exactly the [Duffie & Singleton \[1999\]](#) value of a defaultable claim (intensity κ , discount ι), and it is the *event parametrisation* of the perpetual price of [Ackerer et al. \[2025\]](#): in AHJ’s own funding parametrisation the random-time representation carries the interest parameter in *both* the discount and the effective stopping intensity, $\theta \sim \text{Exp}(\kappa_{\text{AHJ}} - \iota)$ [Theorem 2 of [Ahmed & Bhuyan, 2026, P2](#)], so the two are related by the substitution $\kappa = \kappa_{\text{AHJ}} - \iota$ and coincide identically at $\iota = 0$. Definition 1 deliberately holds the *real-event* intensity κ fixed while the pure time-charge ι varies — the economically meaningful separation experiment, and the one every statement below refers to. In the funding parametrisation, where moving ι also moves the effective intensity, separation is *not* multiplicative for $\iota \neq 0$ (the credit-equivalence paper states this explicitly); nothing in this paper claims otherwise. With a stochastic short rate the factor is $e^{-\int_0^\theta \iota_s ds}$ and all statements below carry over with ι read as the long-run mean.

2 Exact separation

Theorem 1 (Multiplicative separation). *In (1) the interest parameter enters only through the discount functional $D_\iota(\theta) = e^{-\iota\theta}$ and the convergence intensity only through the law of θ . Consequently*

$$f(\iota) = \mathbb{E}^\mathbb{Q}[D_\iota(\theta) X_\theta], \quad D_\iota \equiv 1 \text{ iff } \iota = 0, \quad (2)$$

and the riba-free price is the undiscounted expectation $f(0) = \mathbb{E}^\mathbb{Q}[X_\theta]$, exactly and without remainder.

Proof. $D_\iota(\theta) = e^{-\iota\theta}$ depends on ι alone; the law of θ under Assumption 1 depends on κ alone; X_θ on neither. The integrand factorises as $D_\iota(\theta) \cdot X_\theta$. Setting $\iota = 0$ gives $D_0 \equiv 1$ pointwise, so $f(0) = \mathbb{E}^\mathbb{Q}[X_\theta]$ identically; no limit or approximation is taken. $\square$

Theorem 1 is the algebraic content of separation: interest is a multiplicative overlay on a price whose risk content is carried entirely by θ . The scope is the event parametrisation of Definition 1: in AHJ’s funding parametrisation the same claim would be false for $\iota \neq 0$, because there the interest parameter also shifts the effective stopping intensity — which is precisely the caveat the credit-equivalence paper attaches to its own Separation Theorem [[Ahmed & Bhuyan, 2026, P2](#)]. The two statements are one statement in two coordinate systems, agreeing identically at $\iota = 0$. What Theorem 1 does not yet establish is that $f(0)$ is *finite and non-degenerate*—that removing the overlay does not collapse the price. That is the next, and central, result.

3 The central result: $\iota = 0$ is regular, $r = 0$ is a pole

The classical reason interest-free pricing was thought impossible is that the naive limit is degenerate: a perpetual paying a coupon stream c is worth c/r , which diverges as $r \rightarrow 0$. We show the κ -priced operator does not share this defect, and we identify precisely why.

Theorem 2 (Regularity of the riba-free point). *Suppose $\kappa_t \equiv \kappa > 0$ is constant and X_θ is $\mathbb{Q}$ -integrable. Then:*

- (i) *The map $\iota \mapsto f(\iota)$ extends to a holomorphic function on the half-plane $H_\kappa = \{\iota \in \mathbb{C} : \text{Re}(\iota) > -\kappa\}$, whose only singularity on ∂H_κ is at $\iota = -\kappa$. In particular f is real-analytic on $(-\kappa, \infty)$, the point $\iota = 0$ is interior to H_κ , and the power series of f about 0 has radius of convergence κ .*
- (ii) *In contrast, the conventional perpetual operator $V(r) = \mathbb{E}[\int_0^\infty e^{-rs} c ds] = c/r$ is holomorphic on $\{\text{Re}(r) > 0\}$ with a simple pole at $r = 0$.*
- (iii) *Hence $\iota = 0$ is a regular (removable) point of the κ -priced operator, while $r = 0$ is a non-removable singularity of the conventional operator.*

Proof. (i) Under Assumption 1 with constant κ , $\theta \sim \text{Exp}(\kappa)$. For a bounded payoff, $|X_\theta| \leq M$, dominated convergence gives, for $\text{Re}(\iota) > -\kappa$,

$$f(\iota) = \mathbb{E}[e^{-\iota\theta} X_\theta], \quad |f(\iota)| \leq M \mathbb{E}[e^{-\text{Re}(\iota)\theta}] = M \frac{\kappa}{\kappa + \text{Re}(\iota)} < \infty.$$

The scalar Laplace transform $\mathbb{E}[e^{-\iota\theta}] = \int_0^\infty e^{-\iota t} \kappa e^{-\kappa t} dt = \kappa/(\kappa + \iota)$ is holomorphic on H_κ with a single simple pole at $\iota = -\kappa$; by Morera/dominated differentiation under the expectation, f inherits holomorphy on H_κ , and the distance from 0 to the nearest singularity is κ . For unbounded but integrable X_θ the same conclusion holds on $\{\text{Re}(\iota) \geq 0\}$ by dominated convergence, which already places 0 in the interior of the domain of analyticity. (ii) $\int_0^\infty e^{-rs} c ds = c/r$ for $\text{Re}(r) > 0$, with a simple pole at 0. (iii) Immediate from (i)–(ii): f is analytic at $\iota = 0$ (value $f(0) = \mathbb{E}[X_\theta]$, finite), whereas V has a pole at $r = 0$. $\square$

Remark 1 (The mechanism, in one sentence). The pole of the discounted-expectation operator sits at $\iota = -\kappa$. The convergence intensity $\kappa > 0$ is exactly the margin by which the riba-free point $\iota = 0$ clears that pole. In the conventional perpetual there is no terminating event ($\kappa = 0$), the pole collapses onto the origin, and the discount rate is doing double duty—both discounting and supplying the only convergence—so it cannot be removed. The κ -model separates those two roles: ι discounts, κ converges. Removing ι is harmless precisely because κ , not ι , bounds the horizon ($\theta < \infty$ a.s.).

Remark 2 (Which operator owns the pole). The pole at $\iota = -\kappa$ is a property of the event-parametrised operator (1) — equivalently, of the Duffie-Singleton defaultable claim. In AHJ's funding parametrisation the constant-parameter closed form is $f = \frac{\kappa_{\text{AHJ}} - \iota}{\kappa_{\text{AHJ}}} X$ (at $r_a = r_b$), which is *linear* in ι and has no pole at all: there, raising ι drains the effective event intensity rather than discounting a fixed one, and the price reaches zero at $\iota = \kappa_{\text{AHJ}}$ instead of diverging. The two parametrisations agree on the substantive point — $\iota = 0$ is a finite, regular, no-arbitrage price — and the pole-displacement statement should be read in the event coordinates where it lives. What is common to both, and what the conventional consol lacks, is a terminating event: it is the event, not the coordinates, that makes the riba-free point regular.

Corollary 1 (Smoothness, not singularity, at the riba-free point). *For $\kappa > 0$, f is differentiable to all orders at $\iota = 0$, with $f'(0) = -\mathbb{E}[\theta X_\theta]$ finite; the riba-free price is a regular value approached smoothly from the interest economy, not a singular limit. By contrast $V'(r) = -c/r^2 \rightarrow -\infty$ as $r \downarrow 0$.*

Remark 3 (The margin κ is measured, not assumed). The regularity of Theorem 2 hinges on a single inequality, $\kappa > 0$, which is the margin between the riba-free point and the operator’s pole. That inequality is not a modelling convenience: the convergence intensity is estimated strictly positive in every domain in which it has been measured [Ahmed & Bhuyan, 2026, P15] — $\hat{\kappa} = s/(1-\delta) \approx 0.02$ – 0.2 per year from sukuk and CDS spreads, of order 10^2 – 10^3 per year from the mean-reversion half-life of liquid perpetual bases (hours), and a positive realised claim-arrival rate from insurance-loss records. The pole at $\iota = -\kappa$ therefore sits a strictly positive, empirically bounded distance below zero, and the riba-free price is regular by measurement, not by assumption.

4 Orthogonality of the two parameters

Proposition 1 (Independent axes). *Write the constant-parameter price in closed form, $f(\kappa, \iota) = \frac{\kappa}{\kappa + \iota} \mathbb{E}[X_\theta]$ for a payoff independent of θ . Then κ and ι occupy distinct structural positions: ι enters only the discount factor and κ governs the convergence law, so*

$$\left. \frac{\partial f}{\partial \iota} \right|_{\iota=0} = -\frac{1}{\kappa} \mathbb{E}[X_\theta] \quad (\text{finite}), \quad f(\kappa, 0) = \mathbb{E}[X_\theta] \quad \text{is independent of } \iota,$$

and ι may be set to zero without perturbing the κ -channel. The two are not reparametrisations of a single quantity.

This is the precise sense in which κ “is not r ”: they are different arguments of one pricing function, entering in different slots, with ι removable and κ load-bearing.

5 The Duffie–Singleton isomorphism

Theorem 3 (Formal isomorphism). *Let $\mathcal{A}$ be the perpetual-futures pricing problem $(\kappa, \iota, \theta, X_\theta)$ of Definition 1, and $\mathcal{B}$ the Duffie & Singleton [1999] defaultable-claim problem $(\lambda, r, \tau, R_\tau)$ with default time τ a Cox stopping time of intensity λ and recovery R_τ , priced as $P = \mathbb{E}^\mathbb{Q}[e^{-r\tau} R_\tau]$. The map*

$$\Phi : \kappa \leftrightarrow \lambda, \quad \iota \leftrightarrow r, \quad \theta \leftrightarrow \tau, \quad X_\theta \leftrightarrow R_\tau$$

is a bijection of the two representations under which the no-arbitrage values coincide, $f = P$. Under Φ , $\iota = 0 \leftrightarrow r = 0$, and Theorem 2 transfers verbatim: the defaultable claim at $r = 0$ is regular for $\lambda > 0$. (In AHJ’s funding parametrisation the same map reads $\kappa_{\text{AHJ}} - \iota \leftrightarrow \lambda$, as stated in the credit-equivalence paper [Ahmed & Bhuyan, 2026, P2]; the two coincide at the riba-free point, where both reduce to the headline identification $\kappa \leftrightarrow \lambda$.)

Proof. Both values are expectations under $\mathbb{Q}$ of a discount times a payoff evaluated at the first jump of a Cox process; substituting the four identifications carries (1) to P term by term. Bijectivity is immediate as the substitution is invertible. $\square$

The economic reading: credit risk (λ) and time preference (r) are independent axes that sixty years of reduced-form credit pricing fused into one discounted-hazard expression out of convention. Theorem 3 un-fuses them; Theorem 2 shows the interest axis can be set to zero without collapsing the price.

6 Stochastic intensity and change of measure

We now let κ be stochastic, completing the picture under both measures. Take the Cox–Ingersoll–Ross intensity under the physical measure $\mathbb{P}$,

$$d\kappa_t = \alpha(\bar{\kappa} - \kappa_t) dt + \nu\sqrt{\kappa_t} dW_t^{\mathbb{P}}, \quad 2\alpha\bar{\kappa} \geq \nu^2, \quad (3)$$

so that $\kappa > 0$ a.s. (Feller), keeping Theorem 2 in force pathwise.

Theorem 4 (Q-dynamics, Feller preserved). *Under the completely-affine market price of risk $\Lambda_t = (\lambda/\nu)\sqrt{\kappa_t}$ and the associated Girsanov change to $\mathbb{Q}$, κ remains CIR,*

$$d\kappa_t = \alpha^*(\bar{\kappa}^* - \kappa_t) dt + \nu\sqrt{\kappa_t} dW_t^{\mathbb{Q}}, \quad \alpha^* = \alpha + \lambda, \quad \bar{\kappa}^* = \frac{\alpha\bar{\kappa}}{\alpha + \lambda}, \quad (4)$$

provided $\alpha + \lambda > 0$ (the market price of risk does not overwhelm mean reversion, so that $\alpha^, \bar{\kappa}^* > 0$); then with the same ν , $2\alpha^*\bar{\kappa}^* = 2\alpha\bar{\kappa} \geq \nu^2$ and the Feller condition is preserved.*

Proof. Substitute $dW^{\mathbb{P}} = dW^{\mathbb{Q}} - \Lambda dt$ into (3): the drift becomes $\alpha\bar{\kappa} - (\alpha + \lambda)\kappa$, giving $\alpha^*, \bar{\kappa}^*$ as stated; the product $2\alpha^*\bar{\kappa}^* = 2(\alpha + \lambda)\alpha\bar{\kappa}/(\alpha + \lambda) = 2\alpha\bar{\kappa}$ is invariant. $\square$

7 The affine term structure

The price at t of a unit claim contingent on survival to T (zero recovery), or equivalently the κ -discount factor, is affine in κ .

Proposition 2 (Affine survival price). *Under (4), $P(t, T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T \kappa_s ds} \mid \mathcal{F}_t] = A(\tau) e^{-B(\tau)\kappa_t}$, $\tau = T - t$, where A, B solve the Riccati system*

$$B'(\tau) = 1 - \alpha^* B(\tau) - \frac{1}{2}\nu^2 B(\tau)^2, \quad \frac{A'(\tau)}{A(\tau)} = -\alpha^* \bar{\kappa}^* B(\tau), \quad (5)$$

with $B(0) = 0$, $A(0) = 1$, and closed form, writing $\gamma = \sqrt{(\alpha^)^2 + 2\nu^2}$,*

$$B(\tau) = \frac{2(e^{\gamma\tau} - 1)}{(\gamma + \alpha^*)(e^{\gamma\tau} - 1) + 2\gamma}, \quad A(\tau) = \left[\frac{2\gamma e^{(\alpha^* + \gamma)\tau/2}}{(\gamma + \alpha^*)(e^{\gamma\tau} - 1) + 2\gamma} \right]^{2\alpha^*\bar{\kappa}^*/\nu^2}.$$

Proof. Apply the Feynman–Kac theorem to $P(t, T)$; the PDE $\partial_t P + \alpha^*(\bar{\kappa}^* - \kappa)\partial_{\kappa} P + \frac{1}{2}\nu^2 \kappa \partial_{\kappa\kappa} P - \kappa P = 0$ with the affine ansatz separates into (5); the closed form is Cox et al. [1985]. $\square$

The κ -yield curve is $y(\tau) = -\tau^{-1} \log P(t, T)$, upward-sloping for κ_t below its long-run risk-neutral level. This is the continuous-time object behind the empirically observed, monotone, arbitrage-free κ -surface.

Corollary 2 (No direct rate channel). *Because the price (1) at $\iota = 0$ carries no discount factor in r , a κ -priced claim has no direct interest-rate sensitivity within the pricing formula: $\partial f / \partial r = 0$ at every maturity. Its formula sensitivity is to the convergence intensity alone — an event duration $\partial \log P / \partial \kappa = -(1 - \delta)T$ from the affine form — not to the price of time.*

This is the practical content of separation, stated with its scope. It is a property of the pricing formula, not an observed return: no κ -priced instrument traded through the episode that illustrates it. In the 2022 rate shock the ten-year US Treasury yield rose from $\sim 1\%$ to $\sim 4\%$; a conventional, interest-priced bond lost value in proportion to its duration — from about -4.5% at one year to about -45% at thirty years (a single policy decision revaluing the long end by nearly half) — whereas the $\iota = 0$ formula has no rate term to revalue, by Corollary 2. What it retains is the far shallower event duration of the corollary; the realised credit- κ move over the same period was two to three orders of magnitude smaller than the rate move, so the residual formula loss was 12 to 35 times smaller across the curve (Paper 15, the empirical companion, which also states the equilibrium caveat: in a mixed economy a traded κ -claim competes with r -bearing assets, so its market-clearing intensity κ_Q responds to the rate environment even though the formula's direct rate channel is gone; only a closed κ -world removes the indirect channel as well). The separation of interest from the price is real at the level where pricing formulas operate: it is the difference between carrying a built-in duration gradient and not.

8 Corollary: the riba question as a single parameter

Let $\kappa_P = \mathbb{E}_\infty^\mathbb{P}[\kappa] = \bar{\kappa}$ and $\kappa_Q = \mathbb{E}_\infty^\mathbb{Q}[\kappa] = \bar{\kappa}^*$ be the steady-state physical and risk-neutral intensities, and $\Pi = 1 - \kappa_P/\kappa_Q$ the risk-premium share.

Corollary 3 (Reduction). *Under Theorem 4 with $-\alpha < \lambda < 0$ (so $\alpha^* > 0$ and $\kappa_Q > \kappa_P$), $\Pi = 1 - \bar{\kappa}/\bar{\kappa}^* = |\lambda|/\alpha \in (0, 1)$. After Theorem 2 has removed the predetermined discount at $\iota = 0$, the entire wedge between the priced and the realised intensity is governed by the market price of κ -risk λ , and $\Pi = |\lambda|/\alpha$. The jurisprudential question of whether the residual return is riba in substance is therefore equivalent to a statement about the sign and magnitude of one structural parameter.*

Proof. $\bar{\kappa}^* = \alpha\bar{\kappa}/(\alpha + \lambda)$ gives $\bar{\kappa}/\bar{\kappa}^* = (\alpha + \lambda)/\alpha$, whence $\Pi = 1 - (\alpha + \lambda)/\alpha = -\lambda/\alpha = |\lambda|/\alpha$ for $\lambda < 0$. $\square$

Companion empirical work [Ahmed & Bhuyan, 2026, P15,,] estimates Π directly— $\sim 18\%$ for takaful (small $|\lambda|/\alpha$, the priced intensity tracks realised loss) and the great majority for sovereign credit (large $|\lambda|/\alpha$)—so the analytic reduction here and the measured decomposition there describe one quantity.

9 Application: the money base as a stochastic process

For a κ -priced monetary base $M_t = \sum_i N_i P_i(t)$ with $P_i = \mathcal{P}_i e^{-\beta_i \kappa_t^{(i)} T_i}$, $\beta_i = (1 - \delta_i)$, Proposition 2 and Itô's lemma applied to (4) give

$$dM_t = \underbrace{\sum_i N_i P_i \left[\beta_i T_i \alpha^* (\kappa_t^{(i)} - \bar{\kappa}_i^*) + \frac{1}{2} \beta_i^2 T_i^2 \nu_i^2 \kappa_t^{(i)} \right] dt}_{\text{revaluation drift}} - \underbrace{\sum_i N_i P_i \beta_i T_i \nu_i \sqrt{\kappa_t^{(i)}} dW_t^{(i)}}_{\text{diffusion}} + dM_t^{\text{iss}}. \quad (6)$$

The base contracts when intensities rise ($\partial M / \partial \kappa^{(i)} = -N_i P_i \beta_i T_i < 0$): the countercyclical, debt-free money supply of the κ -Standard, here as the drift of (6).

10 Conclusion

The Separation Theorem is the assertion that interest is a removable parameter of no-arbitrage pricing. We have made it precise. Interest enters the perpetual price as a multiplicative discount overlay on a value carried by a Cox stopping time (Theorem 1); for a strictly positive convergence intensity the operator is analytic at the riba-free point, with its pole displaced to $\iota = -\kappa$, so removing interest yields a finite, regular price where the conventional perpetual diverges (Theorem 2); the two parameters are independent axes (Proposition 1); the construction is isomorphic to the Duffie–Singleton claim (Theorem 3); the intensity carries through a measure change that preserves non-negativity (Theorem 4) and an affine term structure (Proposition 2); and the residual, after interest is gone, is governed by a single risk-price parameter whose magnitude the data measures (Corollary 3). The displacement of the pole off the origin by $\kappa > 0$ is the whole of it: r is the price of time, removable; κ is the price of a real event, load-bearing.

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