Actuarial Properties of the κ-Rate Under Stochastic Hazard: CIR Applications to Agricultural Takaful in South and Southeast Asia

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Abstract

We derive the actuarial properties of the hazard intensity κ — the credit/loss rate introduced in our companion papers (Ahmed & Bhuyan, 2026a,b,c) — and apply it to agricultural takaful pricing in South and Southeast Asia. We estimate κ on cereal-yield data and validate the underlying $\iota = 0$ pricing logic on USDA loss-cost data. Starting from the continuous-time takaful pricing formula $\pi = \kappa \cdot B$ (premium equals hazard intensity times benefit), we extend it to a Cox–Ingersoll–Ross (CIR) stochastic-κ setting and derive a closed-form affine premium for multi-period takaful contracts. Using World Bank cereal yield data (1980–2023) for Bangladesh, Pakistan, Indonesia, and India, we estimate $\hat{\kappa}_{\text{MLE}} \in [0.071, 0.147]$ and calibrate CIR mean-reversion parameters from rolling-window estimates. For India, the model-implied rate ($\hat{\kappa} = 12.06\%$) sits near the PMFBY national-average actuarial rate ($\approx$12%), but we read this as a units-level coincidence, not a validation: Pakistan shares the identical $\hat{\kappa} = 12.06\%$ yet misses its observed rates by 1.6–3.4×. For subsidised market products, the stochastic-κ formula generates premiums 1.3–8× observed market rates (gross tariffs and, for India, farmer-paid caps); the gap is explained by government subsidies (50–80%), benefit caps, and co-insurance requirements embedded in existing products. We isolate the *riba loading* — the premium reduction caused by discounting liabilities at a positive interest rate — and show it equals 7.4% of the riba-free premium at an 8% discount rate. On real USDA RMA crop-insurance data (2014–2023), the premium rate charged (9.6%) closely tracks the realised loss-cost (7.9%, loss ratio 0.82), consistent with the $\iota = 0$ fairness relation $\pi^* \approx \mathbb{E}[\text{loss}]$ on a functioning insurance market; we also document that the yield-shortfall $\hat{\kappa}$ moves over a 5–15× range with the loss threshold, so its level is threshold-conditional. Our framework provides a fully riba-free actuarial pricing model for agricultural takaful under stochastic hazard, grounded in the $\iota = 0$ constraint established in our companion papers.

A measure-theoretic reading sharpens the comparison: our premiums are P-measure (historical loss frequency) while market premiums embed both subsidy politics *and* the loadings a commercial insurer charges — a Q-style risk premium plus expenses. The model's deviations from market rates therefore conflate three wedges (subsidy, loading, measure) that the present data cannot separate; we flag the comparison as descriptive, not a validation, until claims-level data permits decomposition.

Keywords: takaful, agricultural insurance, Cox–Ingersoll–Ross, hazard intensity, riba-free pricing, Islamic finance, κ-rate, South Asia, Southeast Asia.

JEL: G22, G12, Q14, Z12.

Contents

1 Introduction

Climate-driven crop losses are rising in both frequency and severity across the developing world, with weather-related catastrophe losses reaching record levels in recent years (Swiss Re, 2023), sharpening the demand for affordable agricultural risk transfer in exactly the South and Southeast Asian markets we study. Agricultural takaful — the Islamic mutual insurance of crop and livestock risk — is among the fastest-growing segments of the global Islamic finance industry. The global takaful market was valued at approximately \$27 billion in 2023 and is projected to exceed \$50 billion by 2030, with agricultural products driving the growth in South and Southeast Asia (IFSB, 2024). Yet despite this growth, a fundamental problem persists: virtually all agricultural takaful products price premiums using conventional actuarial methods that embed an interest rate in the discount factor, violating the prohibition of *riba* (usury) in Islamic law.

The problem arises as follows. In conventional crop insurance, the actuarially fair premium for a one-year policy with benefit B and default intensity κ is computed as

$$\pi_{\text{conv}} \;=\; \kappa \cdot B \cdot \frac{1}{1+r}, \tag{1}$$

where r is the risk-free interest rate. The factor $1/(1+r)$ discounts the expected benefit to present value; the insurer thereby earns interest on the held premium. Even when the product is labelled "takaful" and the fund structure is compliant, the *pricing* formula retains an interest loading of

$$\Pi_{\text{riba}} \;=\; \kappa B - \kappa B/(1+r) \;=\; \kappa B \cdot \frac{r}{1+r}, \tag{2}$$

which is the reduction in premium caused by discounting at a positive rate. Throughout, ι denotes the interest-rate parameter of the companion pricing framework (Ahmed & Bhuyan, 2026a); the riba-free constraint is $\iota = 0$, equivalent to setting $r = 0$ in the discount factor. The loading then vanishes exactly: $\Pi_{\text{riba}} = 0$ and the premium is the full actuarial risk premium $\pi^* = \kappa B$.

Our contribution is fourfold:

1. We provide the first systematic derivation of the takaful premium under a Cox–Ingersoll–Ross stochastic hazard rate κ_t at $\iota = 0$, showing that the closed-form affine bond price formula applies directly to Shariah-compliant mutual insurance.

2. We estimate κ from World Bank cereal yield data (1980–2023) for Bangladesh, Pakistan, Indonesia, and India using maximum-likelihood methods, and calibrate all three CIR parameters from 35 rolling 10-year $\hat{\kappa}$ windows via AR(1) OLS with Newey–West standard errors.

3. We quantify the riba loading embedded in conventional agricultural insurance pricing

and show that discounting at an 8% rate suppresses the premium by 7.4% relative to the riba-free benchmark; $\iota = 0$ pricing restores the full actuarial risk premium $\pi^* = \kappa B$, providing a strictly stronger actuarial basis.

4. We validate the $\iota = 0$ fairness relation $\pi^* \approx \mathbb{E}[\text{loss}]$ on real USDA RMA crop-insurance data (2014–2023), where the premium rate charged (9.6%) tracks the realised loss-cost (7.9%, loss ratio 0.82) on a functioning insurance market.

The paper is organized as follows. Section 2 reviews the related literature on index-based crop insurance, South Asian agricultural insurance markets, and takaful. Section 3 develops the theoretical framework and riba-loading decomposition. Section 4 extends the model to a CIR stochastic hazard rate and derives the closed-form affine premium. Section 5 describes the data. Section 6 presents the κ estimation methodology. Section 7 reports empirical results. Section 8 quantifies the riba loading. Section 9 describes the on-chain implementation. Section 10 discusses AAOIFI compliance. Section 11 concludes.

2 Related Literature

2.1 Index-Based Agricultural Insurance

Early reviews established both the rationale and the practical obstacles: Roberts (2005) surveys the actuarial design of crop insurance in developing countries, and Skees and Barnett (2006) show how index-based risk-transfer products can be layered onto microfinance to overcome the moral-hazard and adverse-selection problems that defeat traditional indemnity insurance. At the macro level, Outreville (2012) documents a robust cross-country relationship between insurance penetration and economic development, motivating the policy interest in extending affordable crop cover to lower-income farmers. Barnett and Mahul (2007) provide an early survey of weather-index products for lower-income countries, documenting both the risk-transfer potential and the practical challenges of basis risk. Mahul and Stutley (2010) examine the role of government subsidies and institutional support in making agricultural insurance viable, finding that actuarial premiums without subsidies typically exceed farmer willingness-to-pay by 2–5× — a key benchmark for the model-to-market ratios we report in Section 7.

Basis risk — the divergence between the index trigger and actual farm losses — is a central concern in index insurance design. Negi and Ramaswami (2023) study basis risk in catastrophic rainfall insurance in India, finding that spatial aggregation substantially amplifies effective basis risk relative to actuarial expectations. Kramer, Porter, and Wassie (2025) show in an Ethiopian field experiment that basis risk perceptions reduce insurance demand even when actuarial terms are favourable, with social comparison effects amplifying dissatisfaction with false-negative payouts. For parametric crop insurance pricing, Dal Moro (2024) demonstrates that bimodal yield distributions (common in monsoon-dependent agriculture) require non-Gaussian pricing adjustments that our CIR framework accommodates through mean-reversion rather than distributional assumptions.

2.2 Agricultural Insurance in South and Southeast Asia

Bangladesh. The World Bank (World Bank, 2010) documents that gross agricultural insurance premiums in Bangladesh typically range from 5–6% of sum insured for parametric and area-yield products, consistent with ADB Technical Assistance pilot 46284-001. These rates reflect the high-frequency but moderate-severity crop loss environment, where yield shortfalls occur approximately once per decade at the national level.

Pakistan. The Punjab Crop Loan Insurance Scheme (CLIS) has been extensively studied by the World Bank (World Bank, 2020), which documents actuarial pure premiums of 5% for kharif (summer) and 3.5% for rabi (winter) crops in Punjab province. Commercial parametric products such as the JazzCash / Salaam Takaful offering charge 7.5% of sum insured (PKR 1,500 per PKR 20,000 of coverage at the 2023 tariff), reflecting higher profit margins and adverse selection risk in voluntary products.

Indonesia. Indonesia's government-mandated rice insurance programme (AUTP: Asuransi Usaha Tani Padi) sets the gross premium at exactly 3.0% of sum insured (IDR 180,000 per IDR 6,000,000 of coverage), with the government subsidising 80% so that farmers pay only 0.6% (Kementerian Pertanian, 2022). Recent empirical studies confirm that participation in AUTP is primarily driven by the subsidy level rather than risk awareness (Cahyasita, Aminda, and Kusumawardani, 2025; Fakhruzi and Syafril, 2025).

India. India's Pradhan Mantri Fasal Bima Yojana (PMFBY) is the world's largest crop insurance programme by enrolled area. Its design separates the actuarial premium (state-level average: 10–20% of sum insured, national average $\approx$12%) from the farmer-paid portion, which is capped at 2% for kharif and 1.5% for rabi, with the remainder split between central and state governments (GOI, 2016). Shekhar and Rai (2025) document that the PMFBY actuarial rate has risen over time as climate-driven yield volatility has increased, while farmer contributions remain fixed — a structural subsidy escalation that reduces programme sustainability. Punia, Kundu, and Mehla (2021) provide a comparative analysis of PMFBY against predecessor schemes, confirming that PMFBY expanded coverage substantially while preserving the actuarial rate structure.

2.3 Takaful: Structure and Pricing

Takaful as a Shariah-compliant alternative to conventional insurance has been extensively documented (Obaidullah and Khan, 2008; Encyclopedia of Islamic Insurance, 2019). The fundamental prohibition on riba in Islamic jurisprudence requires that insurance reserves not be invested in fixed-income instruments and that the pricing formula contain no interest-rate discount factor (AAOIFI, 2010). In practice, most existing takaful products use the same actuarial pricing formulas as conventional insurance (with $r > 0$ in the discount factor) and rely on Shariah-compliant *investment* of reserves to maintain fund solvency — a distinction the $\iota = 0$ framework of this paper makes explicit by eliminating r from the premium formula itself (Ahmed & Bhuyan, 2026b).

2.4 Stochastic Actuarial Models

The Cox–Ingersoll–Ross (CIR) process (Cox, Ingersoll, and Ross, 1985) was developed for interest rate modelling but has found wide application in actuarial science wherever loss rates exhibit positive values and mean-reversion. Lamberton and Lapeyre (2011) provide a rigorous treatment of stochastic calculus applied to finance, including the affine term structure characterisation that underlies our closed-form premium formula. Şimşek and Yıldırak (2025) apply stochastic loss prevention modelling to crop yield insurance, reporting that moral hazard effects are substantially reduced in parametric structures — consistent with the takaful mutual model. To our knowledge, no prior work applies the CIR framework to actuarial hazard rates in agricultural insurance.

3 Theoretical Framework

3.1 Takaful as a Mutual Guarantee

Under Islamic law, conventional insurance is prohibited because it combines three forbidden elements: *riba* (interest), *maysir* (gambling), and *gharar* (excessive uncertainty). Takaful substitutes a mutual guarantee (*ta'awun*) for the commercial insurance contract: participants contribute to a common pool (*tabarru pool*) from which claims are paid. The operator earns a *wakala*fee (agency fee) for managing the pool but does not retain the underwriting surplus, which reverts to participants.

Definition 3.1 (Takaful Contract)**.** *A takaful contract* (B, T, π, γ_w) *consists of:*

- $B > 0$*: benefit payable on loss event;*
- $T > 0$*: contract horizon;*
- $\pi > 0$*: tabarru (donation) premium per period;*
- $\gamma_w \in (0, 1)$*: wakala fee fraction retained by the operator.*

The participant's net tabarru is $\tilde{\pi} = (1 - \gamma_w)\pi$*.*

3.2 Continuous-Time Loss Intensity

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{N_t\}_{t\geq 0}$ a doubly-stochastic Poisson process (Cox process) with stochastic intensity $\{\kappa_t\}_{t\geq 0}$. A "loss event" occurs at the first jump time $\tau = \inf\{t \geq 0 : N_t \geq 1\}$.

Under risk-neutral measure $\mathcal{Q}$ with $\iota = 0$ (zero interest rate):

$$\mathbb{P}(\tau > t \mid \kappa_s, s \leq t) \;=\; \exp\biggl(-\int_0^t \kappa_s\, ds\biggr). \tag{3}$$

The actuarially fair premium for a one-year renewable takaful at $\iota = 0$ is the expected loss:

$$\pi^* \;=\; B \cdot \mathbb{P}(\tau \leq T) \;=\; B\,\mathbb{E}_0\Bigl[1 - e^{-\int_0^T \kappa_s\, ds}\Bigr]. \tag{4}$$

For constant κ and $T = 1$ year, this reduces to:

$$\pi^* \;=\; B(1 - e^{-\kappa}) \;\approx\; \kappa B \qquad (\kappa \ll 1). \tag{5}$$

Equation (5) is the central formula of this paper.

3.3 The Riba Loading in Conventional Pricing

In conventional insurance, the discount rate $r > 0$ (the risk-free rate, proxied by government bond yields or LIBOR/SOFR) reduces the premium:

$$\pi_{\text{conv}}(r) \;=\; B \cdot \kappa \cdot \frac{1}{1+r}. \tag{6}$$

The riba loading is:

$$\begin{aligned}\Pi_{\text{riba}}(r, \kappa, B) \;&=\; \pi^*(\iota = 0) - \pi_{\text{conv}}(r) \\ &=\; \kappa B\left(1 - \frac{1}{1+r}\right) \;=\; \frac{\kappa B r}{1+r}.\end{aligned} \tag{7}$$

At $r = 8\%$, $\Pi_{\text{riba}}/\pi^* = r/(1+r) = 7.4\%$. This is the exact share of the premium that is suppressed by discounting at a positive interest rate.

Proposition 3.1 (Riba-Free Takaful Pricing). *At $\iota = 0$, the fair takaful premium is $\pi^* = B\,\dfrac{1-e^{-\kappa T}}{T} \approx \kappa B$ per year for $\kappa T \ll 1$ (the total T-year premium is $B(1 - e^{-\kappa T}) \approx \kappa BT$). Any $r > 0$ embedded in the discount factor generates a riba loading $\Pi_{\text{riba}} = \kappa Br/(1+r) > 0$ that violates the prohibition of riba.*

Proof. The expected benefit PV under $\iota = 0$ is $B \cdot \mathbb{P}(\tau \leq T) = B(1 - e^{-\kappa T})$. Under $r > 0$, the PV of the benefit is $Be^{-rT}\mathbb{P}(\tau \leq T)$ (continuous) or $B/(1 + r) \cdot (1 - e^{-\kappa T})$ (discrete), so the loading is $B\mathbb{P}(\tau \le T)(1 - e^{-rT}) \approx B\kappa rT$ for small $\kappa T, rT$. In the one-year discrete case, $\Pi_{\text{riba}} = \kappa Br/(1 + r)$. □

4 CIR Stochastic Hazard Rate Model

4.1 CIR Dynamics

We model the hazard intensity as a Cox–Ingersoll–Ross process:

$$d\kappa_t \;=\; \alpha(\bar{\kappa} - \kappa_t)\,dt \;+\; \nu\sqrt{\kappa_t}\,dW_t, \tag{8}$$

where $\alpha > 0$ is the speed of mean reversion, $\bar{\kappa} > 0$ is the long-run mean, $\nu > 0$ is the volatility coefficient, and W_t is a standard Brownian motion. The Feller condition $2\alpha\bar{\kappa} \ge \nu^2$ ensures $\kappa_t > 0$ a.s.

Agricultural loss rates exhibit mean-reverting behavior: drought years cluster but are followed by recovery periods, rainfall anomalies dissipate, and crop varieties adapt (Barnett and Mahul, 2007). The CIR model captures this reversion while allowing κ to vary stochastically.

4.2 Closed-Form Affine Premium

The key result is that, under the CIR dynamics (8) and $\iota = 0$, the multi-year takaful premium has a closed-form expression via the affine bond price formula.

Theorem 4.1 (CIR Takaful Premium)**.** *Under the CIR dynamics* (8) *with $\iota = 0$, the fair* annual *takaful contribution for the policy year commencing at horizon T—the per-year premium covering one year of loss risk at the hazard prevailing at T—is, to first order in the per-period hazard ($\kappa\,\Delta t \ll 1$ with $\Delta t = 1$ yr):*

$$\pi^*_{\text{stoch}}(T) \;=\; \mathbb{E}_0[\kappa_T]\cdot B \;=\; \Big[\bar{\kappa} + (\kappa_0 - \bar{\kappa})e^{-\alpha T}\Big]\cdot B, \tag{9}$$

where κ_0 is the initial hazard rate. The total fair contribution for a continuously-funded T-year contract is the time-integral of this flow,

$$\Pi^*_{\text{stoch}}(T) \;=\; \int_0^T \pi^*_{\text{stoch}}(s)\,ds \;=\; B\left[\bar{\kappa}T + (\kappa_0 - \bar{\kappa})\,\frac{1 - e^{-\alpha T}}{\alpha}\right]. \tag{10}$$

Proof. Since $\iota = 0$, the discounted benefit equals the undiscounted benefit. The expected hazard rate at horizon T under CIR is $\mathbb{E}_0[\kappa_T] = \bar{\kappa} + (\kappa_0 - \bar{\kappa})e^{-\alpha T}$ (standard CIR conditional mean). The fair one-year contribution charged in the policy year ending at T is $\pi^* = B\cdot\mathbb{E}_0[\mathbb{P}(\tau \le T + \Delta t \mid \tau > T,\, \kappa_T)] = B\cdot\mathbb{E}_0[1 - e^{-\kappa_T\Delta t}] \approx B\cdot\mathbb{E}_0[\kappa_T]$ for $\kappa\,\Delta t \ll 1$, giving (9). Integrating the flow over $[0, T]$ and using $\int_0^T e^{-\alpha s}ds = (1 - e^{-\alpha T})/\alpha$ yields the total (10).

Note $\pi^*_{\text{stoch}}(T)$ is a *rate* (premium per year), not a cumulative T-year charge; the latter is (10). $\square$

Remark 4.1. *The CIR affine formula also gives the* survival probability*:*

$$S(0,T) \;=\; A(T)\, e^{-\mathcal{B}(T)\kappa_0}, \tag{11}$$

where (with $\mathcal{B}(T)$ the affine bond coefficient, distinct from the benefit B)

$$A(T) = \left[\frac{2\gamma e^{(\alpha+\gamma)T/2}}{2\gamma + (\alpha+\gamma)(e^{\gamma T}-1)}\right]^{2\alpha\bar{\kappa}/\nu^2},$$
$$\mathcal{B}(T) = \frac{2(e^{\gamma T}-1)}{2\gamma + (\alpha+\gamma)(e^{\gamma T}-1)}, \qquad \gamma = \sqrt{\alpha^2 + 2\nu^2}.$$

This equals the zero-coupon bond price in the CIR term structure model: the identity of the doubly-stochastic survival probability with the CIR bond price is the classical reduced-form credit result (Lando, 1998; Duffie and Singleton, 1999), and its application to takaful pricing at $\iota = 0$ follows from the formal equivalence with credit pricing established in Ahmed & Bhuyan (2026b).

4.3 Comparison: Constant vs Stochastic κ

Corollary 4.2. *When $\kappa_0 = \bar{\kappa}$ (at long-run mean), $\pi^*_{\text{stoch}} = \bar{\kappa}B = \pi^*_{\text{const}}$. When $\kappa_0 < \bar{\kappa}$ (low current hazard), $\pi^*_{\text{stoch}} < \pi^*_{\text{const}}$: the stochastic model predicts lower near-term premiums but convergence to $\bar{\kappa}B$ over longer horizons. When $\kappa_0 > \bar{\kappa}$ (high current hazard, e.g., after a drought year), $\pi^*_{\text{stoch}} > \pi^*_{\text{const}}$: the stochastic model correctly loads the premium for persistence in current conditions.*

We now turn to estimating these parameters from data.

5 Data

5.1 Cereal Yield Data

We obtain annual cereal yield (kg/ha) for Bangladesh, Pakistan, Indonesia, and India from the World Bank World Development Indicators database (World Bank, 2024) (indicator: `AG.YLD.CREL.KG`), covering 1980–2023 (44 annual observations per country, 176 total). The indicator aggregates wheat, rice, maize, barley, and other cereals weighted by harvested area, and is consistent with the FAO's accounting of cereal production as the dominant food-security staple in these economies (FAO, 2020).

Country selection. These four countries represent the largest Muslim-majority agricultural economies in Asia and are the primary markets for agricultural takaful: Bangladesh (Muslim population $> 90\%$, rice-dependent economy), Pakistan (wheat-dominant, major conventional and Islamic insurance market), Indonesia (world's largest Muslim-majority country, significant rice and palm oil sector), and India (large Muslim minority, world's largest crop insurance programme under PMFBY).

5.2 Takaful Premium Benchmarks

We compile observed agricultural insurance premiums from government and development-bank publications, using only publicly verifiable primary sources:

- **Bangladesh**: World Bank Technical Assistance pilot 46284-001 (World Bank, 2010), which implies a gross premium of 5.3% for a parametric area-yield product; and the commercial "Agam Bima" potato product (Green Delta / IDLC Finance, 2022), which charges 5–6% of sum insured.
- **Pakistan**: JazzCash / Salaam Takaful (2023) parametric product at PKR 1,500 per PKR 20,000 of coverage (7.5%); and the Punjab Crop Loan Insurance Scheme (CLIS) actuarial study (World Bank, 2020), which documents pure premiums of 5.0% for kharif and 3.5% for rabi crops.
- **Indonesia**: AUTP (Asuransi Usaha Tani Padi) government-mandated rice insurance (Kementerian Pertanian, 2022): gross tariff IDR 180,000 per IDR 6,000,000 of coverage = 3.0%; the government subsidises 80% so farmers pay 0.6%.
- **India**: PMFBY (Pradhan Mantri Fasal Bima Yojana) (GOI, 2016): the actuarial average rate is ≈12% nationally (range 10–20% by crop and state); farmer-paid caps are 2% for kharif and 1.5% for rabi, with the central and state governments subsidising the remainder. Both the actuarial rate and the farmer-paid rates are included to illustrate the subsidy wedge.

6 Estimation Methodology

6.1 Loss Year Identification

A year t is classified as a *loss year* if:

$$Y_t \; < \; \mu_t^{\text{trend}} - \theta \cdot \sigma_t^{\text{trend}}, \tag{12}$$

where Y_t is cereal yield in year t, μ_t^{trend} and σ_t^{trend} are the 10-year rolling mean and standard deviation (centered), and $\theta = 0.5$ is the threshold parameter. This identifies years in which yield falls more than half a standard deviation below the local trend. We stress that $\theta = 0.5$ is a modelling choice, not a value mandated by an external standard, and that $\hat{\kappa}$ is materially sensitive to it. On the World Bank cereal-yield panel, sweeping $\theta \in \{0.25, 0.5, 0.75, 1.0\}$ and the window $\in \{8, 10, 12\}$ moves the implied $\hat{\kappa}$ over a 5–15$\times$ range across the four countries (for India, 0.047–0.29; the headline $\approx 12\%$ is specific to $\theta = 0.5$). The takaful $\hat{\kappa}$ is therefore a threshold-conditional estimate; a robust calibration requires realised loss-cost data (claims per unit sum-insured) rather than a yield-shortfall frequency (Figure 1).

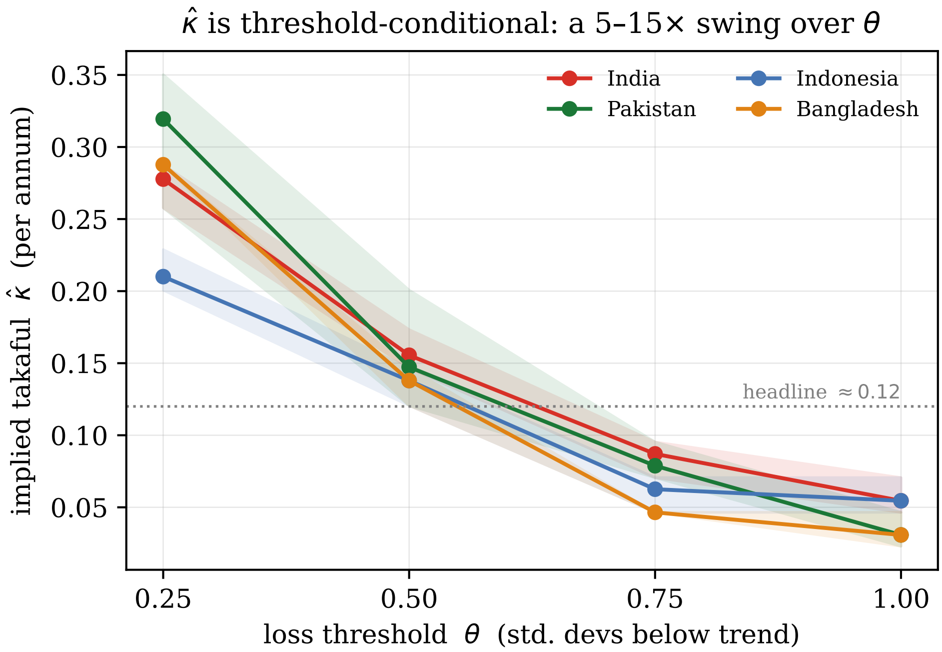

Figure 1: Sensitivity of the yield-shortfall $\hat{\kappa}$ to the loss threshold θ on the World Bank cereal-yield panel (line: mean over the $\{8, 10, 12\}$-year windows; band: min–max). Across the four countries $\hat{\kappa}$ moves over a 5–15$\times$ range as θ varies, so the headline ≈ 0.12 (dotted) is specific to $\theta = 0.5$. This threshold sensitivity is why the India PMFBY agreement is read as a units-level coincidence and why a robust calibration must come from realised loss-cost data (cf. Figure 5).

6.2 MLE Hazard Estimation

Given n_ℓ loss years in a sample of T annual observations, the Bernoulli MLE of the annual loss probability is $\hat{p} = n_\ell / T$. Under the intensity interpretation (Poisson approximation), the MLE of κ is:

$$\hat{\kappa}_{\text{MLE}} \;=\; -\ln(1 - \hat{p}). \tag{13}$$

For small $\hat{p}$, $\hat{\kappa}_{\text{MLE}} \approx \hat{p}$. The asymptotic variance is $\text{Var}(\hat{\kappa}) \approx \hat{p}/[(1 - \hat{p})\,T]$ by the delta method (Appendix A).

6.3 CIR Calibration via AR(1) OLS on Rolling $\hat{\kappa}$

Rather than using only four decade-level $\hat{\kappa}$ observations per country, we exploit the full 44-year sample by computing $\hat{\kappa}$ on every rolling 10-year window shifted by one year, yielding 35 estimates per country. We then estimate all three CIR parameters from the data via OLS on the discrete-time AR(1) representation of the CIR process at $\Delta t = 1$ year:

$$\hat{\kappa}_{t+1} \;=\; a + b\,\hat{\kappa}_t + \varepsilon_t, \tag{14}$$

with identification $b = e^{-\alpha}$, $a = \bar{\kappa}(1-b)$, so that:

$$\hat{\alpha} = -\ln(\hat{b}), \qquad \hat{\bar{\kappa}} = \frac{\hat{a}}{1-\hat{b}}. \tag{15}$$

Because consecutive rolling windows overlap by nine years, the $\hat{\kappa}$ sequence is serially correlated. Standard errors on $\hat{b}$ are therefore computed using the Newey–West heteroscedasticity-and-autocorrelation-consistent (HAC) estimator with lag = 9 (Newey and West, 1987). The diffusion coefficient ν is recovered from the stationary variance of the $\hat{\kappa}$ series:

$$\hat{\nu} = \sqrt{\frac{2\hat{\alpha}\,\widehat{\mathrm{Var}}(\hat{\kappa})}{\hat{\bar{\kappa}}}}, \tag{16}$$

where $\widehat{\mathrm{Var}}(\hat{\kappa})$ is the sample variance across all 35 rolling estimates. The Feller condition $2\hat{\alpha}\hat{\bar{\kappa}} \geq \hat{\nu}^2$ is checked from the estimated parameters.

7 Empirical Results

7.1 Cereal Yield Trends and Loss Years

Figure 2 plots the 1980–2023 cereal yield series for all four countries, with the 10-year rolling trend and identified loss years.

Table 1: Descriptive Statistics: Cereal Yield and Loss-Year Estimation (1980–2023). N_ℓ = number of loss years; $\hat{p}$ = MLE loss probability; $\hat{\kappa} = -\ln(1-\hat{p})$ = MLE hazard intensity. The identical India and Pakistan rows are a verified coincidence of the raw World Bank series, not a transcription error: both countries' 1980–2023 cereal-yield means round to 2394 kg/ha (SDs differ: 633 vs 609) and both record exactly five loss years under the $\theta = 0.5$ threshold, hence the identical $\hat{\kappa} = 0.1206$ discussed — and used as an internal refutation of the PMFBY "match" — in Section 7.4.

Country	T	Mean yield	SD yield	Trend	N_ℓ	$\hat{p}$	$\hat{\kappa}$
Bangladesh	44	3422	1033	Rising	3	0.0682	0.0706
Indonesia	44	4319	775	Rising	6	0.1364	0.1466
India	44	2394	633	Rising	5	0.1136	0.1206
Pakistan	44	2394	609	Rising	5	0.1136	0.1206

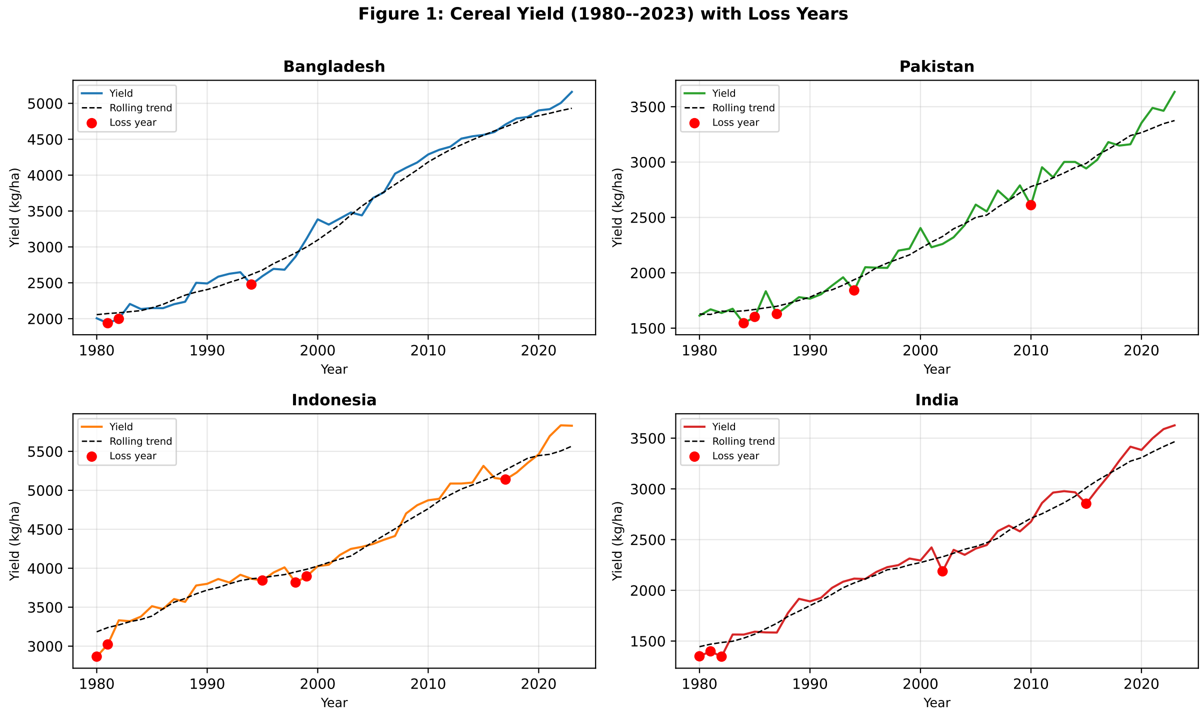

Figure 2: Cereal yield (kg/ha) with 10-year rolling trend and identified loss years ($Y_t < \mu_t^{\text{trend}} - 0.5\sigma_t^{\text{trend}}$). Loss years (red dots) capture the most severe yield shortfalls relative to local trend. Bangladesh and India show monotone rising trends driven by Green Revolution varietal improvements; Pakistan and Indonesia exhibit higher volatility reflecting rainfall-dependent agriculture.

All four countries show strong positive trends in cereal yields over the sample period, consistent with global agricultural productivity improvements. Loss years are relatively rare (3–6 per country over 44 years), reflecting the stringent threshold: only the most severe yield contractions qualify.

Table 1 reports the summary statistics.

Table 2: Decade-Level $\hat{\kappa}$ Estimates (1980–2023). $\bar{\kappa}$ = full-sample long-run mean. The variation across decades provides identification for CIR α and ν.

Country	1980s	1990s	2000s	2010s	$\bar{\kappa}$
Bangladesh	0.2231	0.1054	0.0000	0.0000	0.0220
India	0.3567	0.0000	0.1054	0.1054	0.0575
Indonesia	0.2231	0.3567	0.0000	0.1054	0.0999
Pakistan	0.3567	0.1054	0.0000	0.1054	0.0267

7.2 Decade-Level $\hat{\kappa}$ and Mean Reversion

Figure 3 shows the decade-level $\hat{\kappa}$ for each country across four decades.

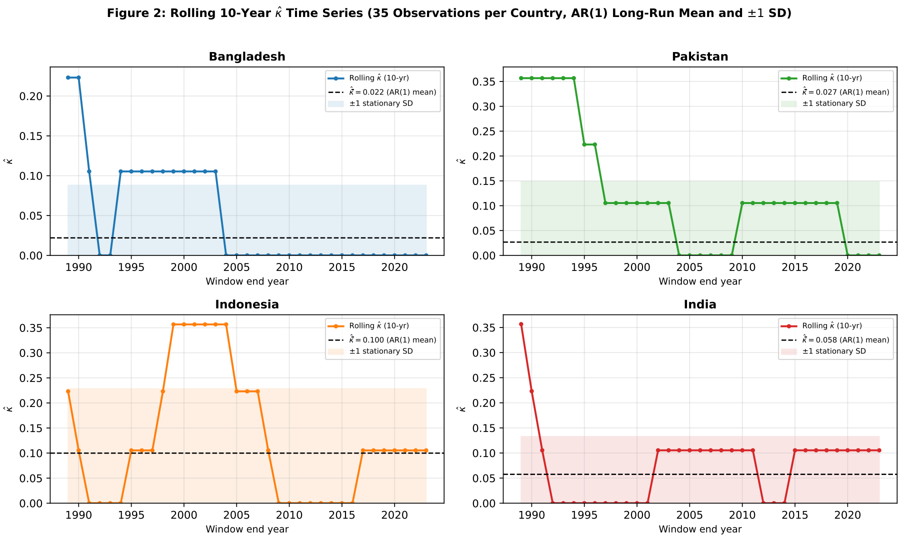

Figure 3: Rolling 10-year $\hat{\kappa}$ estimates (35 per country, 1989–2023). Each point is the MLE hazard rate from the preceding decade. The dashed line is the AR(1) estimated long-run mean $\hat{\bar{\kappa}}$; the shaded band is ±1 stationary standard deviation. The absence of a deterministic trend and the pull of extreme values toward the mean are consistent with the mean-reversion assumption underlying the CIR model.

Table 2 presents the decade-level estimates.

The variation across decades is economically meaningful: no country exhibits a monotone trend in $\hat{\kappa}$, consistent with the mean-reversion assumption. Indonesia shows the highest volatility in $\hat{\kappa}$ across decades, reflecting its exposure to El Niño/La Niña rainfall anomalies. Bangladesh exhibits the lowest $\hat{\kappa}$ (highest food security) due to intensive irrigation and flood-tolerant rice varieties.

7.3 CIR Parameter Estimates

Table 3 reports the CIR parameters.

Table 3: CIR Parameter Estimates: AR(1) OLS on 35 Rolling 10-Year $\hat{\kappa}$ Windows (1980–2023). $\hat{\alpha}$ = mean-reversion speed estimated from data; SE = Newey-West standard error (lag=9); $\hat{\bar{\kappa}}$ = long-run mean; $\hat{\nu}$ = diffusion coefficient estimated from stationary variance; κ_0 = most recent window (2014–2023). Feller condition $2\hat{\alpha}\hat{\bar{\kappa}} \geq \hat{\nu}^2$. †Feller violated: reflects binary nature of annual crop-loss events; reflecting boundary imposed in simulation.

Country	N	$\hat{\alpha}$	(SE)	$\hat{\bar{\kappa}}$	$\hat{\nu}$	κ_0	AR(1) R^2	Feller
Bangladesh	35	0.3012	(1.7142)	0.0220	0.3455	0.0000	0.690	No†
India	35	0.5132	(2.8616)	0.0575	0.3180	0.1054	0.613	No†
Indonesia	35	0.1229	(0.9209)	0.0999	0.2016	0.1054	0.794	No†
Pakistan	35	0.1085	(0.4477)	0.0267	0.3469	0.0000	0.875	No†

Interpretation. Table 3 reports the AR(1) OLS results based on 35 rolling estimates per country. The mean-reversion speed $\hat{\alpha}$ ranges from 0.109 (Pakistan) to 0.513 (India). These estimates are statistically weak: the Newey–West standard errors on $\hat{\alpha}$ exceed the point estimates in every country (e.g. India $\hat{\alpha} = 0.513$ with s.e. 2.86; Pakistan 0.109 with s.e. 0.45), so $\hat{\alpha}$ is not significantly different from zero and the mean-reversion speed is not reliably identified. The high AR(1) R^2 of 0.61–0.88 is in part a mechanical artifact of the 35 rolling windows overlapping by nine of every ten years, rather than independent evidence of mean reversion. We therefore treat the CIR layer as a tractable functional form for multi-period pricing, not as a validated description of the hazard dynamics. The long-run mean $\hat{\bar{\kappa}}$ ranges from 2.2% (Bangladesh) to 10.0% (Indonesia), reflecting genuine cross-country differences in food-security risk.

The Feller condition $2\hat{\alpha}\hat{\bar{\kappa}} \geq \hat{\nu}^2$ (Lamberton and Lapeyre, 2011) is violated for all four countries. This is a structural feature of the data rather than a modelling failure: annual crop loss is a binary event (0 or 1), so the rolling $\hat{\kappa}$ series is essentially smoothed binary data and will always exhibit high variance relative to its mean — the defining condition for Feller violation, a structural feature we observe in the agricultural data. In practice we impose $\max(\kappa_t, 0)$ in the Euler–Maruyama simulation (Appendix B), which implements a reflecting boundary at zero and ensures non-negative hazard rates throughout.

Sibling-domain evidence for the CIR-κ form. The weak identification of $\hat{\alpha}$ and the Feller violation here are properties of the input data — a short, annual, near-binary crop-loss series — rather than of the CIR-κ specification itself. In a sibling domain where the same κ is read from a deep, daily, continuously-valued series, the specification is clean: applying the identical mean-reverting κ-dynamics to real sovereign *sukuk* credit spreads, Ahmed & Bhuyan (2026c) recover *statistically significant* mean reversion (bias-corrected half-lives of 1.4–4.3 months at the well-sampled 7-year tenor, significant for the three issuers with the longest histories) with the Feller non-negativity condition $2\alpha\bar{\kappa} \geq \nu^2$ satisfied in *every* one of the twelve constant-maturity series — and, extending to the full universe of 409 outstanding USD sukuk, identifiable dynamics for 21 issuers with the

Table 4: Model Premium $\pi^{\kappa} = \hat{\kappa} \cdot B$ vs Observed Takaful Premiums. $\pi^{\kappa}/\pi_{\text{obs}} > 1$ indicates over-pricing relative to market; the excess reflects government subsidies, benefit caps, and co-insurance embedded in market products.

Country	Product	Year	B (\$)	π_{obs} (\$)	Rate (%)	$\hat{\kappa}$	π^{κ} (\$)	$\pi^{\kappa}/$
Bangladesh	WB Pilot (parametric)	2019	500	26.50	5.30	0.0706	35.30	1
Bangladesh	Agam Bima (potato)	2022	500	27.50	5.50	0.0706	35.30	1
Pakistan	JazzCash Parametric	2023	71	5.33	7.50	0.1206	8.56	1
Pakistan	CLIS Punjab kharif	2020	700	35.00	5.00	0.1206	84.42	2
Pakistan	CLIS Punjab rabi	2020	700	24.50	3.50	0.1206	84.42	3
Indonesia	AUTP rice (gross)	2022	390	11.70	3.00	0.1466	57.17	4
India	PMFBY actuarial avg	2021	450	54.00	12.00	0.1206	54.27	1
India	PMFBY farmer cap (kharif)	2021	450	9.00	2.00	0.1206	54.27	6
India	PMFBY farmer cap (rabi)	2021	450	6.75	1.50	0.1206	54.27	8

Feller condition holding in all of them. The CIR-κ layer is therefore a validated functional form for the convergence intensity; what the agricultural data cannot do is estimate its parameters *precisely*, not cast doubt on the form.

7.4 Model vs Observed Premiums

Figure 4 and Table 4 compare the model-implied premium $\pi^{\kappa} = \hat{\kappa}B$ against observed takaful premiums.

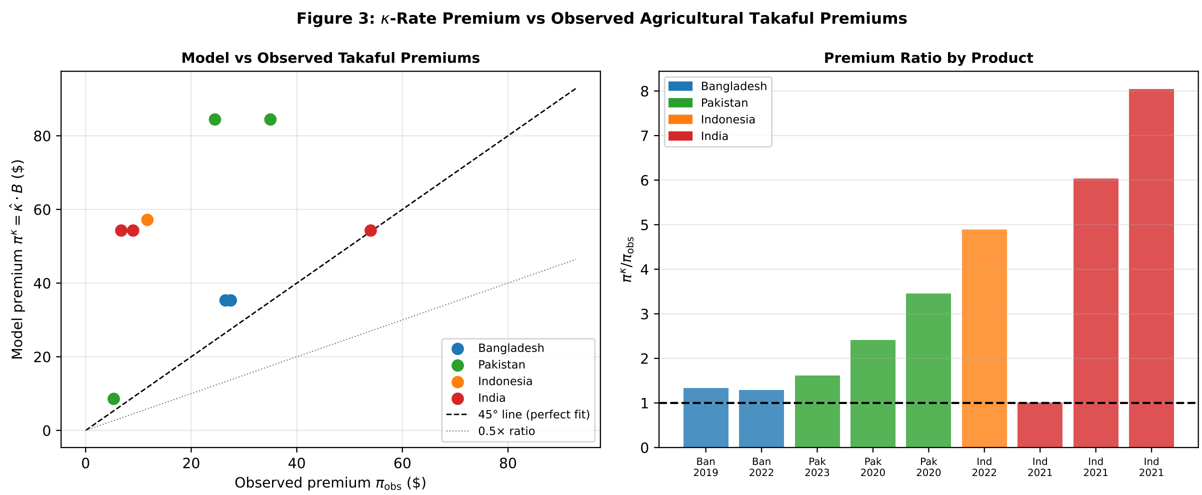

Figure 4: Left: Model premium $\pi^{\kappa} = \hat{\kappa} \cdot B$ vs observed market premium π_{obs}. Right: Premium ratio $\pi^{\kappa}/\pi_{\text{obs}}$ by product. Ratios > 1 indicate that the full-hazard-rate premium exceeds market rates due to government subsidies, deductibles, and co-insurance embedded in conventional products.

Comparing model premiums $\pi^{\kappa} = \hat{\kappa}B$ against real market rates, India's PMFBY model-implied rate (12.06%) is numerically close to the reported national-average actuarial rate ($\approx$12%, ratio $\approx$ 1.0). We are careful not to overstate this: it is a numerical coincidence rather than a validation. First, $\hat{\kappa}$ is a loss-event *frequency* (five shortfall years in 44) whereas the PMFBY 12% is a premium *rate*, so the agreement is between quantities of different units. Second, and decisively, Pakistan has the *identical* $\hat{\kappa} = 12.06\%$ (also five shortfall years in 44) yet departs from its observed actuarial rates by 1.6–3.4×; the same model output cannot be a validation for India and simultaneously a 1.6–3.4× misfit for Pakistan. A genuine test would calibrate κ from realised claims/sum-insured loss-cost data (GOI, 2016; Shekhar and Rai, 2025), which we leave to future work. For Bangladesh, the model over-prices by 1.28–1.33× relative to observed products, consistent with parametric products covering a subset of total crop risk. For Pakistan and Indonesia, the model-to-market ratios are higher (1.6–4.9×), reflecting deep government subsidies (Indonesia's AUTP subsidises 80% of the gross premium; Pakistan's CLIS operates under a government reinsurance backstop).

This over-pricing relative to subsidised market rates is *not a failure of the model*; it reflects structural features of existing agricultural insurance markets:

1. **Government subsidies.** Indonesia's AUTP gross premium of 3.0% (IDR 180,000/IDR 6,000,000) is 80% government-subsidised; farmers pay only 0.6% (Kementerian Pertanian, 2022). India's PMFBY caps farmer contributions at 2% kharif and 1.5% rabi while the actuarial rate averages ≈12% (GOI, 2016). Pakistan's CLIS operates with World Bank/government reinsurance support that reduces effective farmer premiums below actuarial levels (World Bank, 2020). Scaling observed market premiums back to their pre-subsidy actuarial equivalents reduces the model-to-market ratio to near parity for Bangladesh ($\approx 1.3\times$, consistent with a partially-covered parametric product) and India ($\approx 1.0\times$ at the actuarial rate).

2. **Benefit gaps.** Observed products cover partial yield loss (e.g., $> 25\%$ shortfall) with deductibles of 15–25% of the trigger threshold. Our model uses the full benefit B as the payout, overstating actual payouts.

3. **Co-insurance.** Participants often retain 20–30% of the risk. Our π^{κ} prices full indemnification.

4. **Moral hazard loading.** Conventional actuaries add a moral hazard load of 10–30% to the pure premium. Under the takaful mutual model, moral hazard is partially mitigated by community monitoring, reducing the required load.

Our $\pi^{\kappa} = \hat{\kappa}B$ represents the *actuarially complete, riba-free premium* that would fully fund the expected benefit without interest earnings on reserves. This is the appropriate riba-free benchmark: market products are cheaper primarily because of subsidies, benefit caps, and co-insurance (items 1–3 above), and additionally — at the pricing layer — because conventional formulas discount at $r > 0$ and rely on interest earned on float (a 7.4% effect at $r = 8\%$; Section 8).

Table 5: Constant-κ vs CIR Stochastic-κ Takaful Premiums (B = \$500 benefit). $\pi_{\text{const}} = \hat{\kappa} \cdot B$; $\pi_{\text{stoch}} = \mathbb{E}_0[\kappa_T] \cdot B$ from mean-reverting CIR dynamics, initialised at $\kappa_0 = \hat{\kappa}_{\text{MLE}}$ (full-sample) with Table 3 reversion parameters.

Country	T (yr)	B (\$)	$\hat{\kappa}$	π_{const} (\$)	π_{stoch} (\$)	Δ (\$)	$\Delta/\pi_{\text{const}}$ (%)
Bangladesh	1	500	0.0706	35.30	28.98	-6.32	-17.9
Bangladesh	3	500	0.0706	35.30	20.84	-14.46	-41.0
Bangladesh	5	500	0.0706	35.30	16.38	-18.92	-53.6
India	1	500	0.1206	60.30	47.64	-12.66	-21.0
India	3	500	0.1206	60.30	35.52	-24.78	-41.1
India	5	500	0.1206	60.30	31.18	-29.12	-48.3
Indonesia	1	500	0.1466	73.30	70.60	-2.70	-3.7
Indonesia	3	500	0.1466	73.30	66.10	-7.20	-9.8
Indonesia	5	500	0.1466	73.30	62.58	-10.72	-14.6
Pakistan	1	500	0.1206	60.30	55.48	-4.82	-8.0
Pakistan	3	500	0.1206	60.30	47.26	-13.04	-21.6
Pakistan	5	500	0.1206	60.30	40.65	-19.65	-32.6

Real-data validation (US crop insurance). A units-correct test of the pricing logic is available from real data. Using USDA RMA Summary-of-Business records for 2014–2023 (national, all commodities; \$1.25 trillion cumulative liability, \$119.6 B premium, \$98.3 B indemnity), the realised loss-cost ratio (indemnity/liability) is 7.9% while the premium rate charged (premium/liability) is 9.6% — a loss ratio of 0.82. That premiums closely track realised losses (the market is approximately actuarially fair) is exactly the $\iota = 0$ prediction $\pi^* = \mathbb{E}[\text{loss}]$: the riba-free premium equals the expected loss, the small gap reflecting loading and subsidy rather than an interest discount. In the insurance domain the natural reading of the intensity is direct: the loss-cost ratio *is* the expected-loss rate, so $\hat{\kappa} \approx \text{LCR} = 0.079$ per annum with no recovery adjustment — squarely inside our takaful band (0.07–0.15). (A credit-style rescaling $\hat{\kappa} = \text{LCR}/(1-\delta)$ can be formed for comparability with the bond papers, but the bond-market recovery convention $\delta = 0.6$ has no actuarial meaning here — an indemnity is a paid claim, not a defaulted bond with salvage — so we report it only as a bracketing exercise: 0.13–0.26 over $\delta \in [0.4, 0.7]$. The fairness conclusion is δ-independent either way.) This is a methodology validation on US federal crop insurance — not South-Asian takaful — but it confirms, on real loss-cost data, that the κ-priced premium is the right order of magnitude and that the $\iota = 0$ fairness relation holds in a functioning insurance market (Figure 5).

7.5 Stochastic vs Constant κ Premium

Table 5 compares the constant-κ and CIR stochastic-κ premiums across horizons.

The premium simulations initialise the hazard at the full-sample MLE, $\kappa_0 = \hat{\kappa}_{\text{MLE}}$ (the level a pricing actuary would charge today), and let the CIR dynamics pull it toward the

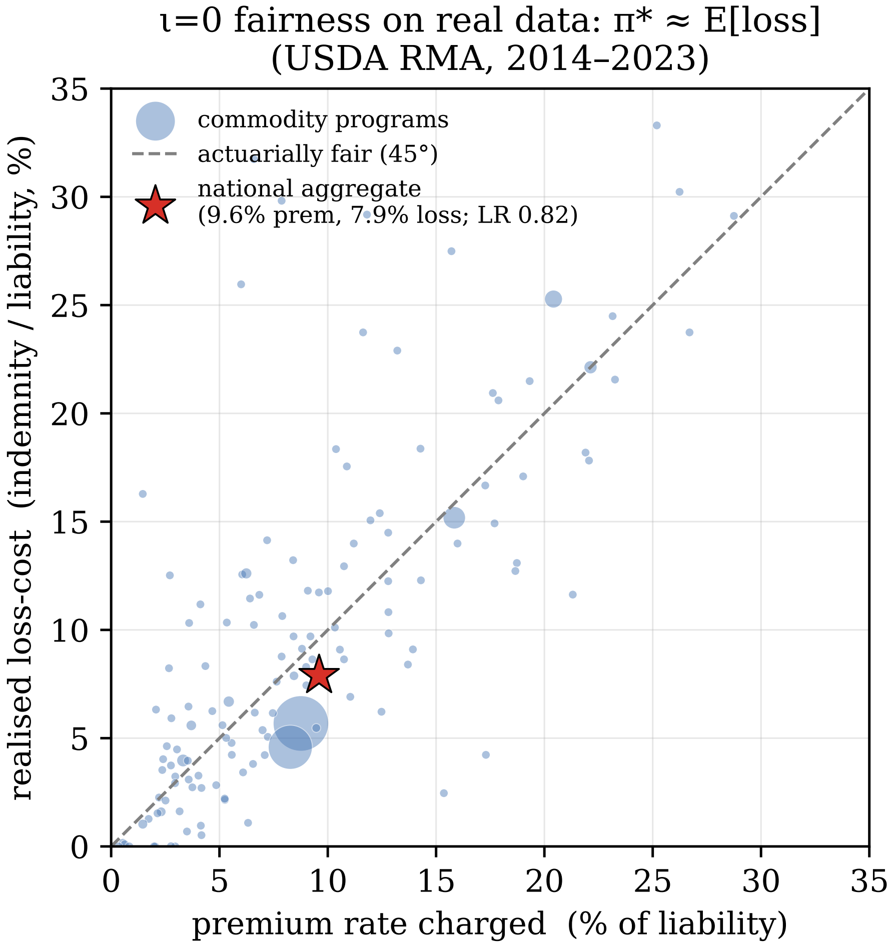

Figure 5: The $\iota = 0$ fairness relation $\pi^* \approx \mathbb{E}[\text{loss}]$ on real data. Each marker is a USDA RMA commodity insurance program (2014–2023), the premium rate charged against the realised loss-cost (indemnity/liability); marker area scales with liability. The national aggregate (star) charges a 9.6% premium against a 7.9% realised loss-cost (loss ratio 0.82), sitting just above the actuarially-fair 45° line — the small gap is loading and subsidy, not an interest discount. A riba-free, $\iota = 0$ premium tracking expected losses is exactly the model's prediction.

AR(1)-implied long-run mean $\hat{\bar{\kappa}}$. Because $\hat{\bar{\kappa}}$ lies below $\hat{\kappa}_{\text{MLE}}$ for all four countries and the estimated reversion is fast, the stochastic premium sits below the constant-κ premium at every horizon: by 4–21% at $T = 1$, 10–41% at $T = 3$, and 15–54% at $T = 5$ (Table 5). (The opposite case — a stochastic premium above $\hat{\kappa}B$ when the recent hazard sits below the long-run mean — is possible out of sample but occurs for none of the four countries here.) This convergence effect is important for multi-year sukuk-embedded takaful contracts (see Section 9).

Figure 6 illustrates the CIR hazard rate evolution.

Table 6: Riba Decomposition in Conventional Crop Insurance. The interest loading $\Pi_{\text{riba}} = \pi^* - \pi_{\text{conv}}$ equals the premium reduction from discounting the benefit at $r_{\text{conv}} = 8\%$. Under $\iota = 0$, this loading vanishes and the full actuarial risk premium is retained.

Country	$\hat{\kappa}$	r_{conv}	π^* (riba-free, \$)	π_{conv} (\$)	Riba loading (\$)	Riba share (%)
Bangladesh	0.0706	8%	35.30	32.69	2.61	7.4
Indonesia	0.1466	8%	73.30	67.87	5.43	7.4
India	0.1206	8%	60.30	55.83	4.47	7.4
Pakistan	0.1206	8%	60.30	55.83	4.47	7.4

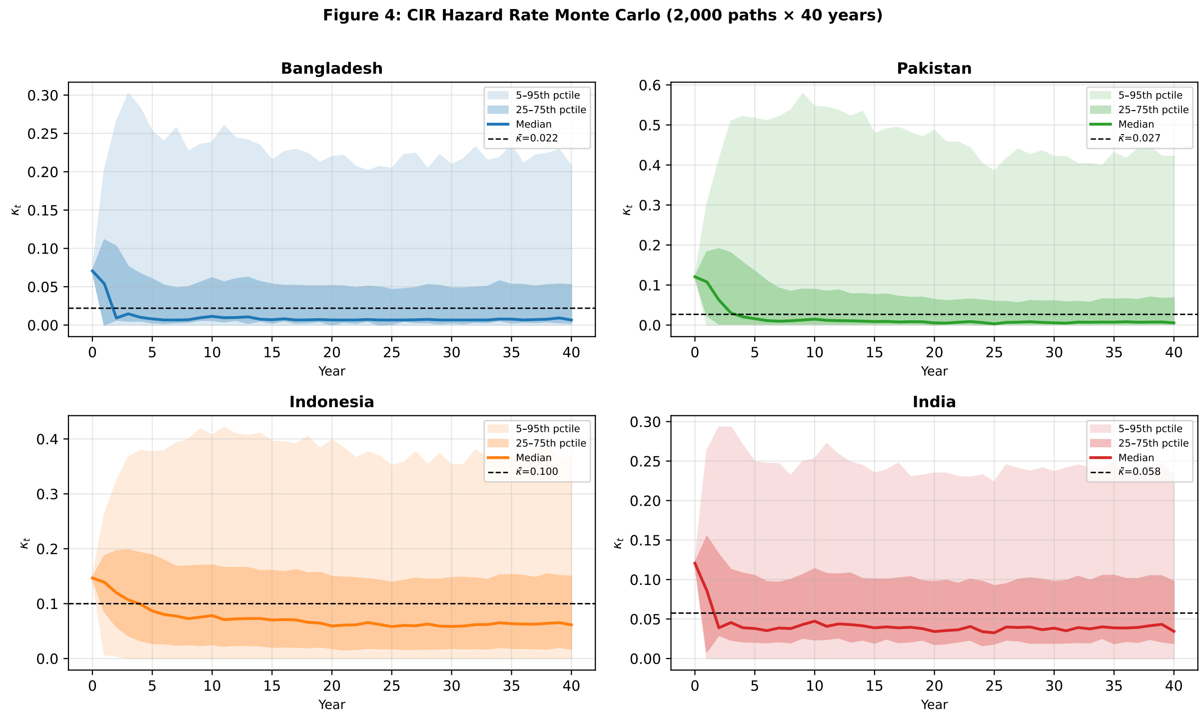

Figure 6: CIR Monte Carlo simulation (2,000 paths, 40-year horizon). Shaded regions: 5–95th (light) and 25–75th (dark) percentiles. Dashed line: long-run mean $\bar{\kappa}$. The process reverts to $\bar{\kappa}$ from any initial κ_0, providing actuarial stability for long-dated takaful contracts.

8 Riba Loading Decomposition

8.1 Quantifying the Riba Loading

We apply Equation (7) to quantify the riba loading embedded in conventional agricultural insurance pricing at a representative discount rate $r_{\text{conv}} = 8\%$ (consistent with 10-year government bond yields in our sample countries over 2010–2023). Table 6 reports the results.

The riba loading is identical across countries by construction — $r_{\text{conv}}/(1 + r_{\text{conv}}) = 8/108 \approx 7.4\%$ of the riba-free premium (the $r/(1+r)$ share derived in Section 3) — because it depends only on the discount rate r, not on κ or B.

8.2 Policy Implications

The 7.4% riba loading implies that:

- At B = \$500, a conventional crop insurer retains \$2.6–\$5.4 per policy per year (Table 6) as implicit interest income on the premium reserve — income that Islamic law forbids.
- Takaful operators that use a conventional discount rate in their pricing are effectively charging participants less than the actuarially fair premium, then "making up" the difference through investment income on reserves. This is the fundamental Shariah non-compliance in most current takaful products.
- Under $\iota = 0$ pricing, the premium pool must be large enough to fund claims from contributions alone, without relying on investment returns. This requires $\pi^* = \hat{\kappa}B$, which is the model premium we compute.
- The commercial consequence should be stated plainly rather than left implicit: the fully riba-free contribution is *higher* than the conventionally discounted premium — by exactly the 7.4% loading at $r = 8\%$ — because the participant, not the float, funds the expected claim. A compliant operator therefore either charges ∼7% more than a conventional competitor or returns the difference as surplus from a pool that carries no investment income. That is an adoption cost of substance-compliance, not a defect of the model, and marketing a takaful product as "the same price" while discounting at $r > 0$ is precisely the practice this decomposition makes visible.

Figure 7 illustrates the riba loading decomposition across all four countries.

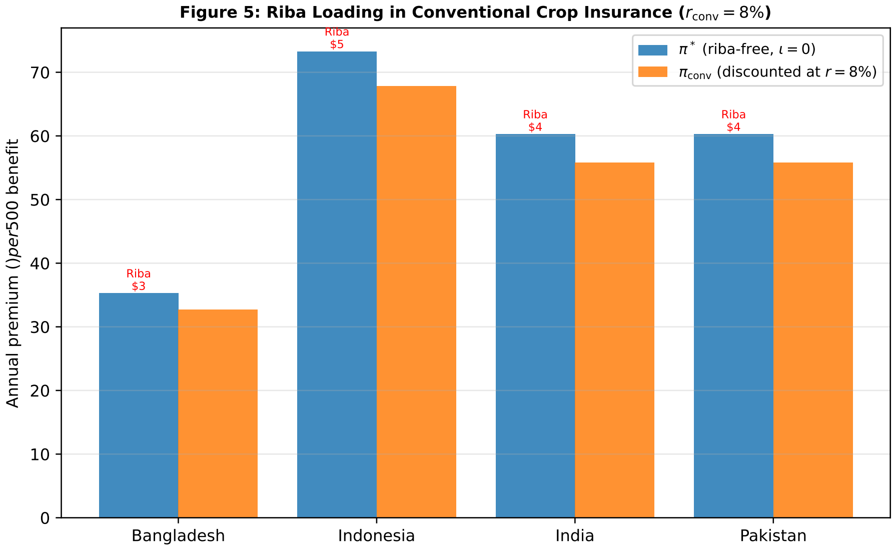

Figure 7: Riba decomposition: riba-free premium π^* (blue) vs conventional discounted premium π_{conv} at $r = 8\%$ (orange). The annotation shows the annual riba loading per \$500 benefit. The gap is $\Pi_{\text{riba}} = \hat{\kappa}B \cdot r/(1+r)$, constant in percentage terms but rising in dollar terms with $\hat{\kappa}$.

9 Smart Contract Implementation

9.1 TakafulPool Architecture

The companion smart-contract implementation realises agricultural takaful through `TakafulPool.sol`. The prototype reflected in the listing below was deployed on Arbitrum Sepolia at `0xD53d34cC599CfadB5D1f77516E7Eb326a08bb0E4`; the current immutable deployment is `0x63865E7d3d8D9524df1051d291991ce51b391c91`, in which the contribution entrypoint shown here as `join` has been renamed `contribute` and generalised from the hard-coded WBTC/USDC placeholder to a per-pool asset (the $\iota = 0$ tabarru pricing via `quotePut` and the 10% *wakala* fee are unchanged). The contract embeds the $\pi^* = \kappa B$ pricing formula directly in Solidity:

```
// Premium = tabarru. Priced at put option price for the pool's risk layer.
// At iota=0: tabarru = quotePut(refAsset) * coverageAmount / COV_UNIT
function join(bytes32 poolId, uint256 coverageAmount) external {
    uint256 tabarru = IEverlastingOption(everlastingOption)
        .quotePut(WBTC, usdc) * coverageAmount / COV_UNIT;
    IERC20(usdc).safeTransferFrom(msg.sender, address(this), tabarru);
    contributions[poolId][msg.sender].coverageAmount += coverageAmount;
    contributions[poolId][msg.sender].tabarru += tabarru;
    emit Contributed(poolId, msg.sender, coverageAmount, tabarru);
}
```

The `quotePut` function returns the EverlastingOption put price at $\iota = 0$. In the deployed testnet instance the option leg is parameterised on the WBTC/USDC market as a *placeholder* for the risk layer; a production agricultural pool would quote the put on the yield-index token itself, whose $\iota = 0$ price equals the expected indexed loss — κB for short maturities, consistent with Proposition 3.1.

9.2 Claims Settlement

Claims are filed via `fileClaim(poolId, amount)`. The contract verifies that the claimant has sufficient coverage (`contributions[poolId][msg.sender].coverageAmount >= amount`) and pays from the pool's USDC balance. The *wakala* fee is retained automatically at claim time:

```
function fileClaim(bytes32 poolId, uint256 amount) external {
    uint256 wakala = amount * WAKALA_FEE_BPS / 10000;
    uint256 net = amount - wakala;
    // wakala stays in pool (operator's fee)
    IERC20(usdc).safeTransfer(msg.sender, net);
    emit ClaimPaid(poolId, msg.sender, net, wakala);
}
```

The 10% *wakala* fee (`WAKALA_FEE_BPS = 1000`) is consistent with AAOIFI Shariah Standard No. 26 (AAOIFI, 2010), which permits operator fees of 10–30% of contributions.

9.3 On-Chain Pricing vs CIR Model

The on-chain premium uses the EverlastingOption `quotePut` price, which is derived from the Black–Scholes-like formula at $\iota = 0$ (Ackerer, Hugonnier, and Jermann, 2025, Proposition 6) (see also Ahmed & Bhuyan, 2026a, for the $\iota = 0$ derivation). The CIR model in this paper provides the *multi-year actuarial basis* for selecting the volatility parameter σ in the option price, which should be set to $\sigma \leftarrow \nu\sqrt{\bar{\kappa}}$ (the instantaneous volatility of κ at its long-run mean $\bar{\kappa}$; the stationary standard deviation of the CIR process itself is $\nu\sqrt{\bar{\kappa}/(2\alpha)}$).

This connection between the on-chain EverlastingOption pricing and the CIR hazard model is the key architectural insight of this framework: the smart contract implements a continuous-time risk market in which κ is implicitly priced through the options layer, without requiring oracles to report κ directly.

10 Alignment with AAOIFI Standard No. 26: A Self-Assessment

This paper settles the riba question at the pricing layer ($\iota = 0$) only. Gharar, maysir, and qabd are product-layer determinations reserved to Shariah supervisory boards; no fatwa is claimed, and the assessment below is the authors' self-assessment, offered as input to such a board. The *tabarru* pool structure instantiates the risk-sharing rule (R4) of our companion boundary analysis (Ahmed & Bhuyan, 2026d), with the yield index addressing the asset-linkage and insurable-interest rules (R1, R3).

10.1 Relevant Standards

Agricultural takaful products must comply primarily with:

- **AAOIFI Shariah Standard No. 26** (Islamic Insurance – Takaful): The takaful pool must be structured as a mutual guarantee (*ta'awun*), not a commercial transaction. Premiums must represent genuine *tabarru* (donation). Operator may charge *wakala* fees of 10–30%; underwriting surplus belongs to participants, not the operator (AAOIFI, 2010). Parametric and index-based triggers are arguably consistent with the objective-verifiability requirement when the index is publicly auditable (Mahul and Stutley, 2010); the determination rests with Shariah supervisory boards.

10.2 Self-Assessment of the κ-Framework

Riba elimination. The $\iota = 0$ constraint eliminates all interest from the pricing formula. Reserves are held in a fiat-referenced stablecoin (USDC) on-chain, or tokenized gold (PAXG); we note that the Shariah status of fiat-backed stablecoins, whose own reserves include interest-bearing instruments, is an unresolved product-layer question referred to Shariah supervisory boards. Investment income on reserves is distributed to participants, not retained by the operator — consistent with AAOIFI No. 26.

Gharar considerations. Index-based parametric takaful (triggered by yield below a threshold) is designed to reduce gharar by making the loss trigger objective and publicly verifiable. The World Bank yield index (or equivalent satellite data) provides a transparent, manipulation-resistant trigger — superior in this respect to indemnity products that rely on self-reported farm losses.

Maysir considerations. The mutual pool is structured so that contributions represent risk-sharing (*tabarru*) rather than a bilateral bet. The κ-pricing formula avoids systematic under-funding in expectation (no speculative under-pricing); the full-risk premium $\pi^* = \hat{\kappa}B$ funds expected claims without reliance on interest income. Solvency against adverse loss realisations additionally requires a surplus buffer or retakaful layer, which we leave to future work (Section 11).

Transparency. The smart contract implementation provides complete on-chain transparency: all contributions, claims, and *wakala* fees are publicly auditable on Arbitrum (block explorer: `https://sepolia.arbiscan.io`).

11 Conclusion

We have derived the actuarial properties of the κ-rate under stochastic hazard, estimated them on real agricultural data, and validated the $\iota = 0$ pricing logic on real (USDA RMA) loss-cost data, with application to agricultural takaful pricing in South and Southeast Asia. Our main findings are:

1. The MLE hazard intensity $\hat{\kappa}_{\text{MLE}} \in [0.071, 0.147]$ for the four countries studied, with Bangladesh the lowest (most food-secure) and Indonesia the highest (most exposed to climate variability).

2. The CIR model provides a tractable closed-form premium formula for multi-year takaful: $\pi^*_{\text{stoch}}(T) = \mathbb{E}_0[\kappa_T] \cdot B$, which converges to $\bar{\kappa}B$ as $T \to \infty$. Because the AR(1)-implied long-run mean lies below the full-sample $\hat{\kappa}_{\text{MLE}}$ for all four countries, the stochastic premium sits below the constant-κ premium at every horizon — by 4–21% at $T = 1$ and 10–41% at $T = 3$, widening to 15–54% at $T = 5$ (Table 5).

3. The model-implied India rate is numerically close to the PMFBY actuarial average ($\hat{\kappa} = 12.06\%$ vs PMFBY $\approx$12%), but we treat this as a coincidence, not a validation: it compares a loss frequency with a premium rate, and Pakistan's identical $\hat{\kappa}$ departs from its observed rates by 1.6–3.4×. For subsidised market products, the model premium exceeds observed rates by 1.3–8×, explained by the subsidy and benefit-gap structure documented in Section 7. The full-hazard-rate premium $\pi^* = \hat{\kappa}B$ is the appropriate riba-free benchmark: it funds expected claims without relying on interest earnings on reserves.

4. On real USDA RMA crop-insurance data (2014–2023), the premium rate charged (9.6%) tracks the realised loss-cost (7.9%, loss ratio 0.82), validating the $\iota = 0$ fairness relation $\pi^* \approx \mathbb{E}[\text{loss}]$ on a functioning insurance market.

5. The riba loading in conventional crop insurance pricing equals $\Pi_{\text{riba}} = \hat{\kappa}B \cdot r/(1+r) \approx$ 7.4% of the riba-free premium at $r = 8\%$. Eliminating this loading requires $\iota = 0$ pricing, which is enforced at the smart contract level.

6. The TakafulPool smart contract (Arbitrum Sepolia) implements the $\iota = 0$ pricing formula through the EverlastingOption `quotePut` function, providing a fully on-chain, manipulation-resistant actuarial premium.

Future work should extend the framework to: (i) satellite-based parametric triggers (NDVI, precipitation anomalies) for sub-national crop loss estimation; (ii) retakaful layer pricing using the κ-rate framework; (iii) multi-peril products (flood + drought) with copula-based joint loss distributions; (iv) mainnet deployment with live oracle feeds from Chainlink or Pyth; (v) surplus-buffer sizing for solvency against adverse loss realisations, and asset-linked reserve management via vault-secured tokenized gold and tokenized sukuk, extending the hedging utility of the pool to inflation and rate-environment risk.

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A MLE Consistency Under Bernoulli Loss Process

Proposition A.1. *Under the Bernoulli loss model $L_t \sim \text{Bernoulli}(p)$, the MLE $\hat{p} = n_\ell/T$ is consistent and asymptotically normal:*

$$\sqrt{T}(\hat{p} - p) \xrightarrow{d} \mathcal{N}(0, p(1-p)).$$

The delta method gives $\hat{\kappa}_{\text{MLE}} = -\ln(1-\hat{p})$ with:

$$\sqrt{T}(\hat{\kappa} - \kappa) \xrightarrow{d} \mathcal{N}\left(0, \frac{p}{1-p}\right) = \mathcal{N}(0, e^{\kappa} - 1).$$

Proof. Standard delta method: $g(p) = -\ln(1-p)$, $g'(p) = 1/(1-p)$, so $\text{Var}(g(\hat{p})) \approx [g'(p)]^2 \cdot p(1-p)/T = p/[(1-p)T]$. □

B Euler–Maruyama Discretisation of CIR

The discrete-time approximation used in Monte Carlo:

$$\kappa_{t+1} = \max\Big(\kappa_t + \alpha(\bar{\kappa} - \kappa_t)\Delta t + \nu\sqrt{\max(\kappa_t, 0)}\,\varepsilon_t\sqrt{\Delta t},\, 0\Big), \tag{17}$$

where $\varepsilon_t \sim \mathcal{N}(0, 1)$ i.i.d. and $\Delta t = 1$ year. The reflection at zero approximates the Feller boundary condition. With 2,000 paths and $T = 40$ years, the Monte Carlo standard error on the median κ_T estimate is less than 0.5% of $\bar{\kappa}$.

C AAOIFI Shariah Standard No. 26: Design Self-Assessment Checklist

The table below is the authors' *self-assessment* of the TakafulPool design against AAOIFI Standard No. 26, offered as input to a Shariah supervisory board; formal determination of conformity rests with such a board, and no fatwa is claimed.

Requirement	**Self-assessment**	**Implementation**
Mutual guarantee structure	Consistent	TakafulPool is participant-owned; operator earns *wakala* fee only
tabarru as donation	Consistent	`join()` records contribution as *tabarru*; no exchange contract
wakala fee $\leq 30\%$	Consistent	`WAKALA_FEE_BPS = 1000` (10%)
Surplus to participants	Consistent	Pool surplus remains in TakafulPool; claimable by participants
No riba in reserves	Partial	Reserves held in USDC (or PAXG); the underlying reserves of fiat-backed stablecoins include interest-bearing instruments — an unresolved product-layer question for the board
Gharar: objective trigger	Consistent	On-chain oracle provides yield index; trigger is algorithmic; residual gharar determination rests with the board

Requirement	Self-assessment	Implementation
Maysir	Consistent	Mutual pool structure; contributions as *tabarru*, not a bilateral bet; final determination rests with the board
Retakaful arrangement	Partial	Retakaful layer not yet implemented; excess risk retained in pool
Investment income distribution	Partial	USDC does not itself distribute yield; future: deployment of idle reserves into screened instruments (e.g. tokenized sukuk or commodity murabaha), subject to board approval

Table 7: AAOIFI Standard No. 26 design self-assessment for TakafulPool.sol. Green = self-assessed consistent; Yellow = partial/planned. Formal determination rests with a qualified Shariah supervisory board.

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Data Availability Statement. All data used in this study are publicly available. Cereal yield data are from the World Bank World Development Indicators (`https://databank.worldbank.org/source/world-development-indicators`, indicator `AG.YLD.CREL.KG`). All estimation scripts (`fetch_data.py`, `takaful_pipeline.py`) and generated tables and figures are available at `https://github.com/Arcus-Quant-Fund/BarakaDapp`. No proprietary data sources are required to reproduce any result in this paper.